# Chapter 1

# Vazane extremy

V teto sekci obdrzime funkci $f$ a mnozinu $M$ a budeme hledat maximum (supremum) a minimum (infimum) funkce $f$ vzhledem k mnozine $M$. Budeme pouzivat vetu (V14) o nabyvani extremu (spojita funkce na kompaktu nabyva maxima i minima), vetu (V16) nutnou podminku existence extremu na otevrene mnozine (nulovost gradientu) a vety (V24), (V25) o Lagrangeove (Lagrangeovych) multiplikatorech. Chtel bych jeste zminit nasledujici uzitecnou poznamku.

**Poznamka 1.1.** *Necht $n \in \mathbb{N}$, $f : \mathbb{R}^n \to \mathbb{R}$ a $M \subset \mathbb{R}^n$. Jestlize $f$ je spojita, pak $\sup_M(f) = \sup_{\overline{M}}(f)$ a $\inf_M(f) = \inf_{\overline{M}}(f)$.*

*Proof.* Protoze $M \subset \overline{M}$, tak $\sup_M(f) \leq \sup_{\overline{M}}(f)$. Oznacime $\sup_{\overline{M}}(f) =: A \in \mathbb{R}^*$. Opacnou nerovnost dokazeme tak, ze pro kazde $B < A$ nalezneme bod $\boldsymbol{x} \in M$, ze $f(\boldsymbol{x}) > B$.

Z definice suprema plyne, ze existuje $\boldsymbol{y} \in \overline{M}$, ze $f(\boldsymbol{y}) > B$. Pokud $\boldsymbol{y} \in M$ jsme hotovi. Predpokladejme, ze $\boldsymbol{y} \notin M$. Pak $\boldsymbol{y} \in \mathrm{bd}(M)$. Ze spojitosti $f$ plyne existence $\delta > 0$, ze $\forall z \in B(\boldsymbol{y}, \delta) : \ f(z) > B$. Z definice $\mathrm{bd}(M)$ plyne, ze existuje $\boldsymbol{x} \in B(\boldsymbol{z}, \delta) \cap M$ a tedy $f(\boldsymbol{x}) > B$. Rovnost pro infimum lze dokazat podobne nebo lze prejit k $-f$.

Rovnez slo dokazat nasledovne. $G := \{\boldsymbol{x} \in \mathbb{R}^n;\ f(x) > B\}$. Z $f$ spojita plyne $G$ otevrena. Z definice suprema plyne $G \cap \overline{M} \neq \emptyset$. Z Vety **??**(xviii) plyne $G \cap M \neq \emptyset$. $\square$

### 1.0.1 Elementarni metody

Nejprve si uvedeme nekolik elementarnich metod, jak hledat extremy funkci.

- Kouknu a vidim (jednoduche odhady),
- slozitejsi odhady,
- parametrizace,
- extremalni body.

#### Kouknu a vidim

- [1. zadani] $f(x, y, z) := x^2 + y^2 + z^2 + 2x + 4y - 6z$, $M := \mathbb{R}^3$.
- [1. reseni] $f(x, y, z) = (x+1)^2 + (y+2)^2 + (z-3)^2 - 14$ tedy $\min_M(f) = -14$ a nabyva se v bode $[-1, -2, 3]$. Z $f(n, 0, 0) = n^2 + 2n \overset{n\to\infty}{\longrightarrow} +\infty$ plyne, ze $\sup_M(f) = +\infty$.

- [2. zadani] $f(x,y,z) := (x+y)^2 + (x-y)^2 + z$, $M := \langle -1,1\rangle^3$.
- [2. reseni] $f(x,y,z) = 2x^2 + 2y^2 + z$. Vsimneme si, ze $\{[x,y] \in \mathbb{R}^2;\ [x,y,z_0] \in M\}$ (rez mnozinou $M$ kolmy na osu $z$ protinajici osu $z$ v hodnote $z_0$) je stejna pro vsechny $z_0 \in \langle -1,1\rangle$. Podobne to je i pro ostatni promenne. Z toho plyne, ze
$$\max_M(f) = \max_{\langle -1,1\rangle} 2x^2 + \max_{\langle -1,1\rangle} 2y^2 + \max_{\langle -1,1\rangle} z = 5$$
a nabyva se v bodech $[\pm 1, \pm 1, 1]$. Obdobne
$$\min_M(f) = \min_{\langle -1,1\rangle} 2x^2 + \min_{\langle -1,1\rangle} 2y^2 + \min_{\langle -1,1\rangle} z = -1$$
a nabyva se v bode $[0,0,-1]$.

**Slozitejsi odhady**

- [1. zadani] $f(x,y) := (x^2+5y^2)e^{-(3x^2+y^2)}$, $M := \mathbb{R}^2$.
- [1. reseni] Vidime, ze $0 = f(0,0) < f(x,y)$ pro $[x,y] \neq [0,0]$. Tedy $\min_M(f) = 0$ a nabyva se v bode $[0,0]$.
$$\nabla f(x,y) = \left[(2x - 6x^3 - 30xy^2)e^{-(3x^2+y^2)}, (10y - 10y^3 - 2x^2y)e^{-(3x^2+y^2)}\right].$$
Z (V16) plyne, ze $f$ muze nabyvat maxima jen v bode, kde $\nabla f(x,y) = [0,0]$. Resime tedy tuto rovnici. Pokud $x = 0$, tak $y \in \{0, \pm 1\}$. Pokud $y = 0$ tak $x \in \left\{0, \pm\frac{1}{\sqrt{3}}\right\}$. Pokud $x, y \neq 0$, tak obdrzime soustavu rovnic
$$\begin{aligned} 6x^2 + 30y^2 &= 2, \\ 2x^2 + 10y^2 &= 10, \end{aligned}$$
ktera nema zadne reseni. Dosadime nalezene body: $f(0,\pm 1) = \frac{5}{e}$ a $f\left(\pm\frac{1}{\sqrt{3}}, 0\right) = \frac{1}{3e}$. Tedy kandidatem na maximum je hodnota $\frac{5}{e}$ nabyvana v bodech $[0,\pm 1]$. Dale ukazeme, ze existuje $R > 0$, ze $\forall [x,y] \notin B(\boldsymbol{o}, R):\ \ f(x,y) < \frac{5}{e}$. Z toho plyne, ze $\sup_M(f) = \sup_{\overline{B(\boldsymbol{o},R)}}(f)$. Protoze $f$ je spojita a $\overline{B(\boldsymbol{o},R)}$ kompaktni (omezena, uzavrena -triv.), tak $f$ nabyva maxima a to $\frac{5}{e}$ v bodech $[0,\pm 1]$. Oznacime $t := x^2 + y^2 \geq 0$. Pak $\varphi(t) := 5te^{-t} \geq f(x,y)$. Dale $\varphi'(t) = 5(1-t)e^{-t} < 0$ pro $t > 1$ a $\varphi(1) = \frac{5}{e}$. Staci tedy volit $R > 1$, pak pro $x^2 + y^2 = t \geq R > 1$ plati, ze
$$f(x,y) \leq \varphi(t) \leq \varphi(R) < \varphi(1) = \frac{5}{e}.$$
- [2. zadani] $f(x,y) := (x+y)e^{-2x-3y}$, $M := \{[x,y] \in \mathbb{R}^2;\ x > 0, y > 0\}$.
- [2. reseni] Ocividne $[0,0] \in \overline{M}$ a $\forall [x,y] \in M:\ \ 0 = f(0,0) < f(x,y)$. Z Poz. 1.1 plyne, ze $\inf_f(M) = 0$ a minimum se nenabyva. Ocividne $\overline{M} = \{[x,y] \in \mathbb{R}^2;\ x \geq 0, y \geq 0\}$. Oznacme $t = x + y \geq 0$ a $\varphi(t) = te^{-2t} \geq f(x,y)$. $\varphi'(t) = (1-2t)e^{-2t} < 0$ pro $t > \frac{1}{2}$. Tedy pro $x + y = t > 1$ plati $f(x,y) \leq \varphi(t) < \varphi(1) = e^{-2} = f(1,0)$. Oznacme $N := \{[x,y] \in \mathbb{R}^2;\ x \geq 0, y \geq 0, x + y \leq 1\}$. Pak $N$ je kompakt (omezena - trojuhelnik, uzavrena). Ze spojitosti $f$ plyne, ze $\sup_{\overline{M}}(f) = \sup_N(f) = \max_N(f) = \max_{\overline{M}}(f)$.
$$\nabla f(x,y) = \left[(1 - 2x - 2y)e^{-2x-3y}, (1 - 3x - 3y)e^{-2x-3y}\right] \neq [0,0].$$
Z (V16) plyne, ze $f$ nenabyva extremu na mnozine $M$. Tedy $f$ nabyva extremu na hranici $M$ (osy $x, y$). $f(x,0) = \varphi(x)$ a $\varphi(x)$ nabyva maxima $\frac{1}{2e}$ v bode $\frac{1}{2}$. Obdobne $f(0,y)$ nabyva maxima $\frac{1}{3e}$ v bode $\frac{1}{3}$. Tedy $\max_{\overline{M}}(f) = \sup_M(f) = \frac{1}{2e}$.

**Parametrizace**

- [1. zadani] $f(x,y) := x^2 + y^2$, $M := \{[x,y] \in \mathbb{R}^2;\ x^2 + 4y^2 = 1\}$.
- [1. reseni] Dosadime do $f$ vztah $x^2 = 1 - 4y^2$ a obdrzime $g(y) := 1 - 3y^2$. Protoze $4y^2 = 1 - x^2 \leq 1$ mame $y \in \left\langle -\frac{1}{2}, \frac{1}{2} \right\rangle$. Funkce $g$ nabyva na intervalu $\left\langle -\frac{1}{2}, \frac{1}{2} \right\rangle$ maximum v bode $y = 0$ a minimum v bodech $y = \pm\frac{1}{2}$. Tedy $f$ nabyva maximum 1 v bodech $[\pm 1, 0]$ a minimum $\frac{1}{4}$ v bodech $\left[0, \pm\frac{1}{2}\right]$.
- [2. zadani] $f(x,y) := x^2 + 12xy + 2y^2$, $M := \{[x,y] \in \mathbb{R}^2;\ 4x^2 + y^2 = 1\}$.
- [2. reseni] Dosadime $y = \pm\sqrt{1-4x^2}$ a obdrzime $g_\pm(x) := f(x, \pm\sqrt{1-4x^2})$. Protoze $x^2 = \frac{1-y^2}{2}$, tak mame $x \in \left\langle -\frac{1}{2}, \frac{1}{2} \right\rangle =: N$. Staci tedy nalezt extremy funkce $g_\pm$ na mnozine $N$. Body podezrele z extremu jsou krajni body a body, kde $g'_\pm(x) = 0$.

$$\begin{aligned}
g_\pm(x) =& 2 - 7x^2 \pm 12x\sqrt{1-4x^2} \implies \\
g'_\pm(x) =& -14x \pm 12\left(\sqrt{1-4x^2} - \frac{4x^2}{\sqrt{1-4x^2}}\right) = 0 \implies \\
(14x\sqrt{1-4x^2})^2 =& 144(1-8x^2)^2 \implies 0 = 10000x^4 - 2500x^2 + 144 \implies \\
0 =& (100x^2)^2 - 25(100x^2) + 144 \implies 100x^2 = \begin{cases} 16 \\ 9 \end{cases} \implies \\
x \in& \left\{\pm\frac{2}{5}, \pm\frac{3}{10}\right\} \implies [x,y] \in \left\{\left[\pm\frac{2}{5}, \pm\frac{3}{5}\right], \left[\pm\frac{3}{10}, \pm\frac{4}{5}\right]\right\}. \\
f\left(\pm\frac{2}{5}, \pm\frac{3}{5}\right) =& \frac{22 \pm 72}{25},\ f\left(\pm\frac{3}{10}, \pm\frac{4}{5}\right) = \frac{137 \pm 288}{100},\ f\left(\pm\frac{1}{2}, 0\right) = \frac{1}{4}.
\end{aligned}$$

Minimum funkce $f$ je tedy $-2$ a nabyva se v bodech $\pm\left[\frac{2}{5}, -\frac{3}{5}\right]$ a maximum je $\frac{17}{4}$ a nabyva se v bodech $\pm\left[\frac{3}{10}, \frac{4}{5}\right]$.

**Extremalni body**

**Definice 1.2.** *Necht $n \in \mathbb{N}$ a $A \subset \mathbb{R}^n$ konvexni. Rekneme, ze bod $\boldsymbol{x}$ je extremalnim bodem mnoziny $A$ ($x \in \operatorname{ext}(A)$), jestlize neexistuje usecka prochazejici bodem $\boldsymbol{x}$ s koncovymi body $\boldsymbol{y}, \boldsymbol{z} \in A$, kde $\boldsymbol{x}, \boldsymbol{y}$ a $\boldsymbol{z}$ jsou navzajem ruzne body.*

**Veta 1.3.** *Necht $n \in \mathbb{N}$, $A \subset \mathbb{R}^n$ konvexni a funkce $f : \mathbb{R}^n \to \mathbb{R}$ nabyva extremu na $A$.*

*(a) Necht $f$ je ryze konvexni. Pak body maxima funkce $f$ vzhledem k mnozine $A$ jsou extremalni body. ($\max_A(f) = \max_{\operatorname{ext}(A)}(f)$)*

*(b) Necht $f$ je ryze konkavni. Pak body minima funkce $f$ vzhledem k mnozine $A$ jsou extremalni body. ($\min_A(f) = \min_{\operatorname{ext}(A)}(f)$)*

*(c) Necht $f$ je konvexni. Pak $\max_A(f) = \max_{\operatorname{ext}(A)}(f)$ a mnozina bodu maxima funkce $f$ vzhledem k mnozine $A$ je podmnozinou konvexniho obalu mnoziny bodu maxima funkce $f$ vzhledem k mnozine $\operatorname{ext}(A)$.*

*(d) Necht $f$ je konkavni. Pak $\min_A(f) = \min_{\operatorname{ext}(A)}(f)$ a mnozina bodu minima funkce $f$ vzhledem k mnozine $A$ je podmnozinou konvexniho obalu mnoziny bodu minima funkce $f$ vzhledem k mnozine $\operatorname{ext}(A)$.*

(e) *Necht $f$ je afinni (konvexni a konkavni). Pak $\max_A(f) = \max_{\mathrm{ext}(A)}(f)$ a mnozina bodu maxima funkce $f$ vzhledem k mnozine $A$ je konvexnim obalem mnoziny bodu maxima funkce $f$ vzhledem k mnozine $\mathrm{ext}(A)$. Obdobne tvrzeni plati pro minima.*

- [1. zadani] $f(x, y) := x + 2y - 3z$, $M := \langle 0, 1\rangle^3$.
- [1. reseni] Mnozina $M$ je krychle a $\mathrm{ext}(M)$ jsou vrcholy teto krychle. Z Vety 1.3 (e) plyne, ze staci vysetrit $\mathrm{ext}(M)$. Maximum 3 tedy funkce $f$ nabyva v bode $[1, 1, 0]$ a minimum $-3$ v bode $[0, 0, 1]$.
- [2. zadani] $f(x, y) := 5x-7y$, $M := \{[x, y] \in \mathbb{R}^2;\ y \geq 0,\ x+y \leq 2,\ x-y \geq 0\}$.
- [2. reseni] Mnozina $M$ je trojuhelnik a $\mathrm{ext}(M) = \{[0, 0], [1, 1], [2, 0]\}$ jsou vrcholy tohoto trojuhelniku. Z Vety 1.3 (e) plyne, ze staci vysetrit $\mathrm{ext}(M)$. Maximum 10 tedy funkce $f$ nabyva v bode $[2, 0]$ a minimum $-2$ v bode $[1, 1]$.

### 1.0.2 Lagrangeovy multiplikatory

#### Postup

Uvedeme si zde postup jak pomoci vety o Lagrangeovych multiplikatorech (V24), (V25) nalezt supremum (maximum) a infimum (minimum) $\mathcal{C}^1$ funkce $f$ vzhledem k omezene mnozine $M$ a pripadne body, ve kterych se techto hodnot nabyva.

(A) Zadefinujeme funkce, ktere budeme dale pouzivat (typicky $\Phi$, pripadne $\Psi$) a overime, ze spolu s funkci $f$ jsou $\mathcal{C}^1$ a tedy i spojite na nejake otevrene nadmnozine mnoziny, kde jsou pouzivany (typicky $\overline{M}$). Casto trivialni. Spocitame gradienty techto funkci. To nam umoznuje v kroku (B) pouzit vetu o urovnovych mnozinach (V11), v kroku (D) vetu o nabyvani extremu (V14) a v krocich ($\mathrm{F}_i$) vety o nutne podmince existence extremu (V16) a o Lagrangeovych multiplikatorech (V24), (V25).

(B) Zjistime, zda mnozina $M$ je uzavrena. Pouzivame zde casto vetu (V11) o urovnovych mnozinach spojite funkce. Pokud ne, tak v bodech (B) - (G) pracujeme misto mnoziny $M$ s mnozinou $\overline{M}$.

(C) Overime, ze mnozina $M$ je omezena. V jednoduchych pripadech muzeme mnozinu nacrtnout nebo slovne popsat (polokoule, kvadr, elipsa, trojuhelnik). Ve slozitejsich pripadech je treba omezenost dokazat pomoci vhodnych odhadu. Nekdy lze pouzit polarni souradnice.

(D) Z (B) a (C) plyne, ze $M$ je kompakt a z (A) tedy plyne dle (V14), ze $f$ nabyva maxima a minima na $M$.

(E) Nyni si rozlozime mnozinu $M = \bigcup M_i$, kde mnoziny $M_i$ budou jednoho z tri nasledujicich typu:

  ($\alpha$) $M_i$ je otevrena.

  ($\beta$) Existuje $\Phi \in \mathcal{C}^1(G)$, ze $M_i = G \cap \Phi^{-1}(\{0\})$, kde $G$ je nejaka otevrena mnozina.

  ($\gamma$) Existuje $\Phi, \Psi \in \mathcal{C}^1(G)$, ze $M_i = G \cap \Phi^{-1}(\{0\}) \cap \Psi^{-1}(\{0\})$, kde $G$ je nejaka otevrena mnozina.

($\mathrm{F}_i$) Nyni pro kazdou mnozinu $M_i$ pomoci vet (V16), (V24), (V25) nalezneme vsechny body v teto mnozine, kde muze byt extrem (budeme znacit cervene). Temito body jsou body, kde nasledujici vektory jsou linearne zavisle: V pripade ($\alpha$) je to $\nabla f(\boldsymbol{x})$, v pripade ($\beta$) je to $\nabla f(\boldsymbol{x})$, $\nabla\Phi(\boldsymbol{x})$ a v pripade ($\gamma$) je to $\nabla f(\boldsymbol{x})$, $\nabla\Phi(\boldsymbol{x})$ a $\nabla\Psi(\boldsymbol{x})$. Predpoklady vet jsou splneny, neb dle (A) jsou dane funkce tridy $\mathcal{C}^1$. Presny postup je nasledujici:

($\alpha$) Dle (V16) resim $\nabla f(\boldsymbol{x}) = \boldsymbol{o}$,

($\beta$) Pouzivam (V24) pro vazbu $\Phi$.

  (a) Hledam $\boldsymbol{x} \in M_i$, ze $\nabla\Phi(\boldsymbol{x}) = \boldsymbol{o}$.

  (b) Resim nasledujici soustavu rovnic:

$$\begin{aligned} \nabla f(\boldsymbol{x}) + \lambda\nabla\Phi(\boldsymbol{x}) =& \boldsymbol{o}, \\ \Phi(\boldsymbol{x}) =& 0, \end{aligned}$$

  kde $\lambda$ je nejake realne cislo.

($\gamma$) Pouzivam (V25) pro vazby $\Phi$ a $\Psi$.

  (a) Hledam $\boldsymbol{x} \in M_i$, ze $\nabla\Psi(\boldsymbol{x}) = 0$ nebo existuje $\alpha \in \mathbb{R}$, ze

$$\alpha\nabla\Psi(\boldsymbol{x}) = \nabla\Phi(\boldsymbol{x}).$$

  (b) Resim nasledujici soustavu rovnic:

$$\begin{aligned} \nabla f(\boldsymbol{x}) + \lambda\nabla\Phi(\boldsymbol{x}) + \mu\nabla\Psi(\boldsymbol{x}) =& \boldsymbol{o}, \\ \Phi(\boldsymbol{x}) =& 0, \\ \Psi(\boldsymbol{x}) =& 0, \end{aligned}$$

  kde $\lambda, \mu$ jsou nejaka realna cisla.

(G) Dosadime do funkce $f$ vsechny body nalezeny v krocich ($\mathrm{F}_i$). Dle (D) vime, ze alespon jeden z techto bodu je bodem maxima (a dle (V16), (V24), (V25) mimo tyto body maximum byt nemuze) a alespon jeden z techto bodu je bodem minima (a dle (V16), (V24), (V25) mimo tyto body minimum byt nemuze). Body, ve kterych je nejvyssi hodnota jsou tedy vsechny body maxima a body s nejnizsi hodnotou jsou vsechny body minima.

(H) Pokud puvodni mnozina $M$ neni uzavrena (viz bod (B)), tak zjistim, zda nektere body maxima lezi v puvodni $M$. Pokud ano, tak toto jsou skutecne body maxima. Pokud ne, tak se maxima nenabyva a nalezena hodnota je dle Poznamky 1.1 hodnotou suprema puvodni mnoziny $M$. Podobny proces se provede i pro minimum (resp. infimum).

### 1. priklad

Zadani: $f(x,y,z) := 10z + x - y$, $M := \{[x,y,z] \in \mathbb{R}^3;\ x^2+y^2+z^2 \leq 1,\ x+y \geq 0\}$.

Reseni:

(A) Oznacime $\Phi(x,y,z) := x^2+y^2+z^2-1$, $\Psi(x,y,z) := x+y$. $f, \Phi, \Psi$ jsou polynomy a tedy $f, \Phi, \Psi \in \mathcal{C}^1(\mathbb{R}^3)$.

$$\nabla f(x,y,z) = [1,-1,10], \quad \nabla\Phi(x,y,z) = [2x,2y,2z], \quad \nabla\Psi(x,y,z) = [1,1,0].$$

(B) $M = \Phi^{-1}((-\infty,0\rangle) \cap \Psi^{-1}(\langle 0,+\infty))$, $\Phi, \Psi \in \mathcal{C} \overset{(V11)}{\Longrightarrow} M$ je uzavrena.

(C) $M$ je jednotkova polokoule a tedy omezena.

(D) OK

(E) $M = M_1 \cup M_2 \cup M_3 \cup M_4$, kde

$$\begin{aligned} M_1 :=& \Phi^{-1}((-\infty,0)) \cap \Psi^{-1}((0,+\infty)), && \text{typ } (\alpha),\\ M_2 :=& \Phi^{-1}(\{0\}) \cap \Psi^{-1}((0,+\infty), && \text{typ } (\beta),\\ M_3 :=& \Phi^{-1}((-\infty,0)) \cap \Psi^{-1}(\{0\}), && \text{typ } (\beta),\\ M_4 :=& \Phi^{-1}(\{0\}) \cap \Psi^{-1}(\{0\}), && \text{typ } (\gamma). \end{aligned}$$

($F_1$) $\nabla f(x,y,z) \neq \boldsymbol{o}$.

($F_2$) Lag. mult. (V24) pro vazbu $\Phi$.

(a) $\nabla\Phi(x,y,z) = \boldsymbol{o} \Leftrightarrow [x,y,z] = \boldsymbol{o} \notin M_2$.

(b)

$$\begin{aligned} (I):&& 1+2\lambda x =& 0, & (I) \Rightarrow \lambda \neq& 0,\\ (II):&& -1+2\lambda y =& 0, & (I)+(II): 2\lambda(x+y) =& 0,\\ (III):&& 10+2\lambda z =& 0, & x+y =& 0,\\ (IV):&& x^2+y^2+z^2 =& 1. \end{aligned}$$

Protoze $\Psi(x,y,z) = x+y > 0$ na mnozine $M_2$, tak reseni rovnic v mnozine $M_2$ neexistuje.

($F_3$) Lag. mult. (V24) pro vazbu $\Psi$.

(a) $\nabla\Psi(x,y,z) \neq \boldsymbol{o}$.

(b) $\nabla f(x,y,z) + \lambda\nabla\Psi(x,y,z) = [1+\lambda, -1+\lambda, 10] \neq \boldsymbol{o}$.

($F_4$) Lag. mult. (V25) pro vazby $\Phi$ a $\Psi$.

(a) $\nabla\Psi(x,y,z) \neq \boldsymbol{o}$. Rovnice

$$\alpha\nabla\Psi(x,y,z) = [\alpha,\alpha,0] = [2x,2y,2z] = \nabla\Phi(x,y,z)$$

je splnena pro nejake $\alpha \in \mathbb{R}$ pokud $x = y$ a $z = 0$. Z $\Psi(x,x,0) = 2x = 0$ na $M_4$ plyne, ze $x = y = 0$. Bod $[0,0,0] \notin M_4$.

(b)

$$\begin{aligned} (I):&& 1+2\lambda x+\mu =& 0, & (I)+(II): 2\lambda(x+y)+2\mu =& 0,\\ (II):&& -1+2\lambda y+\mu =& 0, & (IV) \Rightarrow \mu = 0,\ (III) \Rightarrow \lambda \neq& 0,\\ (III):&& 10+2\lambda z =& 0, & 10(II)+(III): 10y+z =& 0,\\ (IV):&& x+y =& 0, & (IV),\ (V) \Rightarrow 102y^2 =& 1,\\ (V):&& x^2+y^2+z^2 =& 1, & \pm\sqrt{\frac{1}{102}}[-1,1,-10] =& [x,y,z]. \end{aligned}$$

(G) $f\left(\pm\sqrt{\frac{1}{102}}[-1,1,-10]\right) = \mp\sqrt{102}$.

(H) Funkce $f$ nabyva na mnozine $M$ minimum $-\sqrt{102}$ pouze v bode $\sqrt{\frac{1}{102}}[-1,1,-10]$ a maximum $\sqrt{102}$ pouze v bode $\sqrt{\frac{1}{102}}[1,-1,10]$.

**2. priklad**

Zadani: $f(x,y,z) := 2x - 3y + z$, $M := \{[x,y,z] \in \mathbb{R}^3;\ x^2+y^2+z^2 < 1,\ 2x + y - z \leq 0\}$.

Reseni:

(A) Oznacime $\Phi(x,y,z) := x^2+y^2+z^2-1$, $\Psi(x,y,z) := 2x+y-z$. $f, \Phi, \Psi$ jsou polynomy a tedy $f, \Phi, \Psi \in \mathcal{C}^1(\mathbb{R}^3)$.

$$\nabla f(x,y,z) = [2,-3,1], \quad \nabla\Phi(x,y,z) = [2x,2y,2z], \quad \nabla\Psi(x,y,z) = [2,1,-1].$$

(B) Mnozina $M$ neni uzavrena. $\overline{M} = \Phi^{-1}((-\infty,0\rangle) \cap \Psi^{-1}((-\infty,0\rangle)$.

(C) $\overline{M}$ je jednotkova polokoule a tedy omezena.

(D) OK

(E) $\overline{M} = M_1 \cup M_2 \cup M_3 \cup M_4$, kde

$$\begin{aligned}
M_1 &:= \Phi^{-1}((-\infty,0)) \cap \Psi^{-1}((-\infty,0)), && \text{typ } (\alpha),\\
M_2 &:= \Phi^{-1}(\{0\}) \cap \Psi^{-1}((-\infty,0)), && \text{typ } (\beta),\\
M_3 &:= \Phi^{-1}((-\infty,0)) \cap \Psi^{-1}(\{0\}), && \text{typ } (\beta),\\
M_4 &:= \Phi^{-1}(\{0\}) \cap \Psi^{-1}(\{0\}), && \text{typ } (\gamma).
\end{aligned}$$

($F_1$) $\nabla f(x,y,z) \neq \boldsymbol{o}$.

($F_2$) Lag. mult. (V24) pro vazbu $\Phi$.

(a) $\nabla\Phi(x,y,z) = \boldsymbol{o} \Leftrightarrow [x,y,z] = \boldsymbol{o} \notin M_2$.

(b)

$$\begin{aligned}
(I):&\quad 2+2\lambda x = 0, & 3(I)+2(II): 2\lambda(3x+2y) &= 0,\\
(II):&\quad -3+2\lambda y = 0, & (II)+3(III): 2\lambda(y+3z) &= 0,\\
(III):&\quad 1+2\lambda z = 0, & (I) \Rightarrow \lambda \neq 0 \Rightarrow 3x+2y &= 0,\ y+3z=0,\\
(IV):&\quad x^2+y^2+z^2 = 1, & z = -\frac{1}{3}y,\ x = -\frac{2}{3}y, (IV) &\Rightarrow \frac{14}{9}y^2 = 1, \Rightarrow
\end{aligned}$$

$$\Longrightarrow y = \pm\frac{3}{\sqrt{14}} \Longrightarrow \pm\sqrt{\frac{1}{14}}\,[-2,3,-1] = [x,y,z].$$

Protoze $\Psi\left(\pm\sqrt{\frac{1}{14}}\,[-2,3,-1]\right) = 0$, tak reseni rovnic v mnozine $M_2$ neexistuje.

($F_3$) Lag. mult. (V24) pro vazbu $\Psi$.

(a) $\nabla\Psi(x,y,z) \neq \boldsymbol{o}$.

(b)

$$\begin{aligned}
(I):&\quad 2+2\lambda = 0, & (II) \Rightarrow \lambda &= 3,\\
(II):&\quad -3+\lambda = 0, & (III) \Rightarrow \lambda &= 1,\\
(III):&\quad 1-\lambda = 0,\\
(IV):&\quad 2x+y-z = 0.
\end{aligned}$$

Tedy reseni neexistuje.

($F_4$) Lag. mult. (V25) pro vazby $\Phi$ a $\Psi$.

(a) $\nabla\Psi(x,y,z) \neq \boldsymbol{o}$. Rovnice

$$\alpha\nabla\Psi(x,y,z) = [2\alpha, \alpha, -\alpha] = [2x, 2y, 2z] = \nabla\Phi(x,y,z)$$

je splnena pro nejake $\alpha \in \mathbb{R}$ pokud $x = 2y$ a $z = -y$. Z $\Psi(2y, y, -y) = 6y = 0$ na $M_4$ plyne $x = y = z = 0$. Bod $[0,0,0] \notin M_4$.

(b)

$$\begin{aligned}
(I): \quad 2 + 2\lambda x + 2\mu &= 0,\\
(II): \quad -3 + 2\lambda y + \mu &= 0,\\
(III): \quad 1 + 2\lambda z - \mu &= 0,\\
(IV): \quad 2x + y - z &= 0,\\
(V): \quad x^2 + y^2 + z^2 &= 1,
\end{aligned}$$

$$\begin{aligned}
(VI): \ 2(I) + (II) - (III): 0 &= 2\lambda(2x + y - z) + 6\mu\\
&= 2\lambda\Psi(x,y,z) + 6\mu\\
&\overset{(IV)}{=} 6\mu.
\end{aligned}$$

Z (VI) plyne, ze $\mu = 0$ a lze tedy pocitat jako v pripade ($F_2$) a obdrzime body $\pm\sqrt{\frac{1}{14}}[-2, 3, -1]$.

(G) $f\left(\pm\sqrt{\frac{1}{14}}[-2,3,-1]\right) = \mp\sqrt{14}$.

(H) Body nalezene v kroku ($F_4$) nejsou v $M$ a tedy funkce $f$ nenabyva na mnozine $M$ minimum, $\inf_M(f) = -\sqrt{14}$ a nenabyva ani maximum, $\sup_M(f) = \sqrt{14}$.

## 3. priklad

Zadani: $f(x,y,z) := e^{xyz}$, $M := \{[x,y,z] \in \mathbb{R}^3;\ x^2 + 2y^2 + 3z^2 = 30\}$.

Reseni: Protoze exponenciela je rostouci, staci hledat extremy funkce $g(x,y,z) := xyz$ a nasledne je dosadit do exponenciely.

(A) Oznacime $\Phi(x,y,z) := x^2 + 2y^2 + 3z^2 - 30$. $g, \Phi$ jsou polynomy a tedy $g, \Phi \in \mathcal{C}^1(\mathbb{R}^3)$.

$$\nabla g(x,y,z) = [yz, xz, xy], \quad \nabla\Phi(x,y,z) = [2x, 4y, 6z].$$

(B) $M = \Phi^{-1}(\{0\})$, $\Phi \in \mathcal{C} \overset{(V11)}{\Longrightarrow} M$ je uzavrena.

(C) $M$ je povrch elipsoidu a tedy omezeny.

(D) OK

(E) Mnozina $M = M_1$ je typu $\beta$.

($F_1$) Lag. mult. (V24) pro vazbu $\Phi$.

(a) $\nabla\Phi(x,y,z) = \boldsymbol{o} \Leftrightarrow [x,y,z] = \boldsymbol{o} \notin M_1$.

(b)

$$\begin{aligned}
(I): \quad yz + 2\lambda x &= 0, & x(I) - y(II): 2\lambda(x^2 - 2y^2) &= 0,\\
(II): \quad xz + 4\lambda y &= 0, & x(I) - z(III): 2\lambda(x^2 - 3z^2) &= 0,\\
(III): \quad xy + 6\lambda z &= 0, & \text{pokud } \lambda \neq 0 \Rightarrow x^2 = 2y^2 &= 3z^2 \Rightarrow\\
(IV): \quad x^2 + 2y^2 + 3z^2 &= 30, & \overset{(IV)}{\Longrightarrow} x = \pm\sqrt{10} &\Rightarrow
\end{aligned}$$

$$\Longrightarrow \left[\pm\sqrt{10}, \pm\sqrt{5}, \pm\sqrt{\frac{10}{3}}\right] = [x, y, z].$$

Pokud $\lambda = 0$, tak $[x, y, z] \in N := \{[0, 0, \pm\sqrt{10}], [0, \pm\sqrt{15}, 0], [\pm\sqrt{30}, 0, 0]\}$.

(G) $g\left(\left[\pm\sqrt{10}, \pm\sqrt{5}, \pm\sqrt{\frac{10}{3}}\right]\right) = \pm 10\sqrt{\frac{5}{3}}$ a $g(N) = \{0\}$.

(H) Funkce $f$ nabyva minimum $e^{-10\sqrt{\frac{5}{3}}}$ na mnozine $M$ pouze v bodech $\left[\pm\sqrt{10}, \pm\sqrt{5}, \pm\sqrt{\frac{10}{3}}\right]$, kde lichy pocet souradnic je zaporny. Funkce $f$ nabyva maximum $e^{10\sqrt{\frac{5}{3}}}$ na mnozine $M$ pouze v bodech $\left[\pm\sqrt{10}, \pm\sqrt{5}, \pm\sqrt{\frac{10}{3}}\right]$, kde sudy pocet souradnic je zaporny.

**4. priklad**

Zadani: $f(x, y, z) := \operatorname{arccotg}(yz)$, $M := \{[x, y, z] \in \mathbb{R}^3;\ x^2+z^2 \leq 4,\ x^2+y^2 \leq 1\}$.

Reseni: Protoze funkce arccotg je klesajici, staci hledat extremy funkce $g(x, y, z) := yz$ a nasledne je dosadit do funkce arccotg.

(A) Oznacime $\Phi(x, y, z) := x^2 + z^2 - 4$, $\Psi(x, y, z) := x^2 + y^2 - 1$. $g, \Phi, \Psi$ jsou polynomy a tedy $g, \Phi, \Psi \in \mathcal{C}^1(\mathbb{R}^3)$.

$$\nabla g(x, y, z) = [0, z, y], \quad \nabla\Phi(x, y, z) = [2x, 0, 2z], \quad \nabla\Psi(x, y, z) = [2x, 2y, 0].$$

(B) $M = \Phi^{-1}((-\infty, 0\rangle) \cap \Psi^{-1}((-\infty, 0\rangle)$, $\Phi, \Psi \in \mathcal{C} \overset{(V11)}{\Longrightarrow} M$ je uzavrena.

(C) $\Psi(x, y, z) \leq 0 \Rightarrow |x|, |y| \leq 1$, $\Phi(x, y, z) \leq 0 \Rightarrow |z| \leq 2$. Tedy $M$ je omezena.

(D) OK

(E) $M = M_1 \cup M_2 \cup M_3 \cup M_4$, kde

$$\begin{aligned}
M_1 &:= \Phi^{-1}((-\infty, 0)) \cap \Psi^{-1}((-\infty, 0)), && \text{typ } (\alpha), \\
M_2 &:= \Phi^{-1}(\{0\}) \cap \Psi^{-1}((-\infty, 0)), && \text{typ } (\beta), \\
M_3 &:= \Phi^{-1}((-\infty, 0)) \cap \Psi^{-1}(\{0\}), && \text{typ } (\beta), \\
M_4 &:= \Phi^{-1}(\{0\}) \cap \Psi^{-1}(\{0\}), && \text{typ } (\gamma).
\end{aligned}$$

($F_1$) $\nabla g(x, y, z) = 0 \Leftrightarrow y = z = 0$. Body $[x, 0, 0] \in M_1 \Leftrightarrow |x| < 1$.

($F_2$) Lag. mult. (V24) pro vazbu $\Phi$.

(a) $\nabla\Phi(x, y, z) = \boldsymbol{o} \Leftrightarrow x = z = 0$. Body $[0, y, 0] \notin M_2$.

(b)

$$\begin{aligned}
(I): &\quad 2\lambda x = 0, && (II), (IV) \Rightarrow x = \pm 2, \\
(II): &\quad z = 0, && [\pm 2, y, 0] \notin M_2, \\
(III): &\quad y + 2\lambda z = 0, \\
(IV): &\quad x^2 + z^2 = 4.
\end{aligned}$$

($F_3$) Lag. mult. (V24) pro vazbu $\Psi$.

(a) $\nabla\Psi(x, y, z) = \boldsymbol{o} \Leftrightarrow x = y = 0$. Body $[0, 0, z] \notin M_3$.

(b)

$$\begin{aligned}
(I):&\quad 2\lambda x =0, &\quad (III),(IV) \Rightarrow x =\pm 1,\\
(II):&\quad z+2\lambda y =0, &\quad (II),(III) \Rightarrow z =0,\\
(III):&\quad y =0, &\quad [x,y,z] =[\pm 1,0,0] \in M_3,\\
(IV):&\quad x^2+y^2 =1.
\end{aligned}$$

($F_4$) Lag. mult. (V25) pro vazby $\Phi$ a $\Psi$.

(a) $\nabla\Psi(x,y,z)=\boldsymbol{o} \Leftrightarrow x=y=0$. Body $[0,0,z]\notin M_2$. Rovnice

$$\alpha\nabla\Psi(x,y,z)=[2\alpha x,2\alpha y,0]=[2x,0,2z]=\nabla\Phi(x,y,z)$$

je splnena pro nejake $\alpha\in\mathbb{R}$ pokud $x=z=0$ nebo $y=z=0$. Body $[0,y,0],[x,0,0]\notin M_4$.

(b)

$$\begin{aligned}
(I):&\quad 2\lambda x+2\mu x =0; &\quad \text{pokud } x=0 \overset{(IV),(V)}{\Longrightarrow} z=\pm 2, y=\pm 1 \Rightarrow\\
(II):&\quad z+2\mu y =0; &\quad \Longrightarrow [x,y,z] =[0,\pm 1,\pm 2],\\
(III):&\quad y+2\lambda z =0; &\quad \text{pokud } x\neq 0 \overset{(I)}{\Longrightarrow} \lambda=-\mu \Rightarrow\\
(IV):&\quad x^2+z^2 =4; &\quad \overset{(VI)}{\Longrightarrow} 2\mu(y^2+z^2) =0,\\
(V):&\quad x^2+y^2 =1; &\quad (V)-(IV): y^2-z^2 =-3,
\end{aligned}$$

$$(VI)=y(II)-z(III):\ 2(\mu y^2-\lambda z^2)=0,$$

Pokud $\mu\neq 0$, tak $y^2+z^2=0$ a tedy $2y^2=-3$, coz nelze. Pokud $\mu=-\lambda=0$, tak z (II) a (III) plyne $y=z=0$. Z (IV) pak plyne $x=\pm 2$ a z (V) plyne $x=\pm 1$, coz nelze.

(G) $g(x,0,0)=0$, $g(0,\pm 1,\pm 2)=\pm 2$.

(H) Funkce $f$ nabyva na mnozine $M$ minimum $\operatorname{arccotg}(2)$ pouze v bodech $\pm[0,1,2]$ a maximum $\operatorname{arccotg}(-2)$ pouze v bodech $\pm[0,1,-2]$.

**5. priklad - domaci ukol**

Zadani: $f(x,y,z):=xy+z$, $M:=\{[x,y,z]\in\mathbb{R}^3;\ x^2+y^2=2,\ xyz\geq 1,\ z\geq 1\}$.

Reseni: Protoze $[1,1,n]\in M$ pro kazde $n\in\mathbb{N}$ a $f(1,1,n)=n+1\to+\infty$, tak $\sup_M(f)=+\infty$. Budeme tedy hledat pouze minimum (infimum). Definujme $g(z):=z+\frac{1}{z}$. Pak $g$ je spojita na $\langle 1,+\infty)$, $g'(z)=1-\frac{1}{z^2}$ a $g'(z)>0$ pro $z\in(1,+\infty)$. Tedy funkce $g$ ma v bode 1 ostre minimum vzhledem k mnozine $\langle 1,+\infty)$. Vime, ze pro $[x,y,z]\in M$ plati $xyz\geq 1$ a $z\geq 1$. Tedy $xy\geq\frac{1}{z}$ a

$$f(x,y,z)\geq z+\frac{1}{z}=g(z)\geq g(1)=2.$$

a rovnosti vyse nastavaji pouze v pripade, ze $z=1$ a $xyz=1$. Pak na $M$ mame $x^2+y^2=2$ a $xy=1$. Z cehoz plyne, ze

$$(x-y)^2=x^2+y^2-2xy=2-2=0\Rightarrow x=y=1.$$

Minimum funkce $f$ na mnozine $M$ je tedy 2 a nabyva se pouze v bode $[1,1,1]$.

**6. priklad**

Zadani: $f(x,y) := e^y$, $M := \{[x,y] \in \mathbb{R}^2;\ (x^2+y^2)^2 = 2(x^2-y^2)\}$.

Reseni: Protoze exponenciela je rostouci, staci hledat extremy funkce $g(x,y) := y$ a nasledne je dosadit do exponenciely.

(A) Oznacime $\Phi(x,y) := (x^2+y^2)^2 - 2(x^2-y^2)$. $g, \Phi$ jsou polynomy a tedy $g, \Phi \in \mathcal{C}^1(\mathbb{R}^2)$.

$$\nabla g(x,y) = [0,1], \quad \nabla\Phi(x,y) = [4x(x^2+y^2-1), 4y(x^2+y^2+1)].$$

(B) $M = \Phi^{-1}(\{0\})$, $\Phi \in \mathcal{C} \overset{(V11)}{\Longrightarrow} M$ je uzavrena.

(C) Pouziji polarni souradnice $x = r\cos(\alpha)$, $y = r\sin(\alpha)$. Pak

$$\Phi(x,y) = (x^2+y^2)^2 - 2(x^2-y^2) = r^4 - 2r^2\cos(2\alpha) = 0.$$

Tedy $r = 0$ nebo $r^2 = 2\cos(2\alpha) \le 2$. Z cehoz plyne, ze $M \subset \overline{B(\boldsymbol{o}, \sqrt{2})}$ a je tedy omezena.

(D) OK

(E) Mnozina $M = M_1$ je typu $\beta$.

($F_1$) Lag. mult. (V24) pro vazbu $\Phi$.

(a) $\nabla\Phi(x,y) = \boldsymbol{o} \Leftrightarrow [x,y] \in \{[\pm 1, 0], [0,0]\}$. Body $[\pm 1, 0] \notin M$.

(b)

$$\begin{aligned} (I):& \qquad \lambda 4x(x^2+y^2-1) = 0, \quad \text{pokud } x = 0 \overset{(III)}{\Longrightarrow} y = 0,\\ (II):& \qquad 1 + \lambda 4y(x^2+y^2+1) = 0, \qquad (II) \Rightarrow \lambda \neq 0,\\ (III):& \qquad (x^2+y^2)^2 - 2(x^2-y^2) = 0, \quad \text{pokud } x \neq 0 \overset{(I)}{\Longrightarrow} x^2+y^2 = 1, \Rightarrow \end{aligned}$$

$$\overset{(III)}{\Longrightarrow} 2(x^2-y^2) = 1 \Rightarrow 4x^2 = 3 \Rightarrow x = \pm\frac{\sqrt{3}}{2},\ y = \pm\frac{1}{2}.$$

Mame tedy $[x,y] = \frac{1}{2}[\pm\sqrt{3}, \pm 1]$.

(G) $g\left(\frac{1}{2}[\pm\sqrt{3}, \pm 1]\right) = \pm\frac{1}{2}$ a $g(0,0) = 0$.

(H) Funkce $f$ nabyva minimum $\frac{1}{\sqrt{e}}$ na mnozine $M$ pouze v bodech $\frac{1}{2}[\pm\sqrt{3}, -1]$. Funkce $f$ nabyva maximum $\sqrt{e}$ na mnozine $M$ pouze v bodech $\frac{1}{2}[\pm\sqrt{3}, 1]$.

**7. priklad**

Zadani: $f(x,y) := \operatorname{arctg}(x)$, $M := \{[x,y] \in \mathbb{R}^2;\ (x^2+y^2)^2 = Kxy\}$, $K > 0$.

Reseni: Protoze arctg je rostouci funkce, staci hledat extremy funkce $g(x,y) := x$ a nasledne je dosadit do funkce arctg.

(A) Oznacime $\Phi(x,y) := (x^2+y^2)^2 - Kxy$. $g, \Phi$ jsou polynomy a tedy $g, \Phi \in \mathcal{C}^1(\mathbb{R}^2)$.

$$\nabla g(x,y) = [1,0], \quad \nabla\Phi(x,y) = [4x(x^2+y^2) - Ky, 4y(x^2+y^2) - Kx].$$

(B) $M = \Phi^{-1}(\{0\})$, $\Phi \in \mathcal{C} \overset{(V11)}{\Longrightarrow} M$ je uzavrena.

(C) Pouziji polarni souradnice $x = r\cos(\alpha)$, $y = r\sin(\alpha)$. Pak

$$\Phi(x,y) = (x^2+y^2)^2 - Kxy = r^4 - \frac{r^2}{2} K \sin(2\alpha) = 0.$$

Tedy $r = 0$ nebo $r^2 = \frac{1}{2} K \sin(2\alpha) \le \frac{K}{2}$. Z cehoz plyne, ze $M \subset \overline{B\left(\boldsymbol{o}, \sqrt{\frac{K}{2}}\right)}$ a je tedy omezena.

(D) OK

(E) Mnozina $M = M_1$ je typu $\beta$.

($F_1$) Lag. mult. (V24) pro vazbu $\Phi$.

(a) Resime $\nabla\Phi(x,y) = \boldsymbol{o}$:

$$\begin{aligned}
4x(x^2+y^2) =& Ky \quad \Rightarrow 4xy(x^2+y^2) = Ky^2,\\
4y(x^2+y^2) =& Kx \quad \Rightarrow 4yx(x^2+y^2) = Kx^2 \Rightarrow\\
x^2 - y^2 =& 0 \qquad\qquad\qquad \Rightarrow x = \pm\, y,\\
\Phi(x, \pm x) = 4x^4 \mp Kx^2 =& 0 \qquad \Leftrightarrow (x = y = 0) \vee \left(x = y = \pm\frac{\sqrt{K}}{2}\right).
\end{aligned}$$

Mame tedy body $[0,0]$ a $\pm\left[\frac{\sqrt{K}}{2}, \frac{\sqrt{K}}{2}\right]$.

(b)

$$\begin{aligned}
(I):&\quad 1 + \lambda(4x(x^2+y^2) - Ky) = 0, &&(I) \Rightarrow \lambda \neq 0,\\
(II):&\quad \lambda(4y(x^2+y^2) - Kx) = 0, &&(III) \Rightarrow x = 0 \Leftrightarrow y = 0,\\
(III):&\quad (x^2+y^2)^2 - Kxy = 0, &&\text{pokud } x, y \neq 0 \Rightarrow\\
\frac{y}{\lambda}(II) - (III):&\quad (x^2+y^2)(3y^2 - x^2) = 0 &&\stackrel{x,y\neq 0}{\Longrightarrow} x^2 = 3y^2, \Rightarrow
\end{aligned}$$

$$\Phi(\pm\sqrt{3}y, y) = 16y^4 \mp K\sqrt{3}y^2 = 0 \Rightarrow (y = 0) \vee \left(x = \sqrt{3}y, y = \pm\frac{\sqrt{K\sqrt{3}}}{4}\right).$$

Mame tedy $[x,y] = [0,0]$ nebo $[x,y] = \pm\frac{1}{4}\left[\sqrt{3K\sqrt{3}}, \sqrt{K\sqrt{3}}\right]$.

(G) $g(0,0) = 0$, $g\left(\pm\left[\frac{\sqrt{K}}{2}, \frac{\sqrt{K}}{2}\right]\right) = \pm\frac{\sqrt{K}}{2}$ a

$$g\left(\pm\frac{1}{4}\left[\sqrt{3K\sqrt{3}}, \sqrt{K\sqrt{3}}\right]\right) = \pm\frac{\sqrt{3K\sqrt{3}}}{4}.$$

Navic $\frac{\sqrt{K}}{2} < \frac{\sqrt{3K\sqrt{3}}}{4}$.

(H) Funkce $f$ nabyva minimum $-\frac{\sqrt{3K\sqrt{3}}}{4}$ na mnozine $M$ pouze v bode $\frac{1}{4}\left[-\sqrt{3K\sqrt{3}}, -\sqrt{K\sqrt{3}}\right]$. Funkce $f$ nabyva maximum $\frac{\sqrt{3K\sqrt{3}}}{4}$ na mnozine $M$ pouze v bode $\frac{1}{4}\left[\sqrt{3K\sqrt{3}}, \sqrt{K\sqrt{3}}\right]$.

**8. priklad**

Zadani: $f(x,y) := \frac{x-y}{x^2+y^2+1}$, $M := \mathbb{R}^2$.

Reseni: Uvedomme si, ze $f(x,y) = -f(-x,-y)$. Funkce $f$ ma tedy maximum $m$ v bode $A$ prave tehdy kdyz, ma funkce $f$ minimum $-m$ v bode $-A$. Staci tedy nalezt pouze maximum a body maxima funkce $f$. Uvedomme si

$$|f(x,y)| = \frac{|x-y|}{x^2+y^2+1} \leq \frac{|x|+|y|}{x^2+y^2} \leq \frac{|x|+|y|}{\frac{(|x|+|y|)^2}{2}} = \frac{2}{|x|+|y|}.$$

Pokud tedy $[x,y] \notin N := (-4,4)^2$, tak $f(x,y) \leq \frac{1}{2} < \frac{2}{3} = f(1,-1)$. Protoze $\overline{N}$ je kompakt (omezena a uzavrena) a $f \in \mathcal{C}(\mathbb{R}^2)$, tak $f$ nabyva maxima na $\overline{N}$. Vzhledem k nerovnostem vyse, vsak toto maximum musi byt v $N$ a tedy $\sup_M(f) = \sup_N(f) = \max_N(f) = \max_M(f)$. $N$ je otevrena, $f \in \mathcal{C}^1(\mathbb{R}^2)$ a tak staci nalezt body, kde gradient $f$ je nulovy (viz. (V16)) a z techto bodu vybrat ten s nejvyssi funkcni hodnotou.

$$(I): \quad \frac{\partial f}{\partial x} = \frac{x^2+y^2+1-2x(x-y)}{(x^2+y^2+1)^2} = \frac{y^2+1-x^2+2xy}{(x^2+y^2+1)^2} = 0,$$

$$(II): \quad \frac{\partial f}{\partial y} = \frac{-x^2-y^2-1-2y(x-y)}{(x^2+y^2+1)^2} = \frac{y^2-1-x^2-2xy}{(x^2+y^2+1)^2} = 0,$$

$$(x^2+y^2+1)^2((I)+(II)): \quad 2(y^2-x^2) = 0 \Rightarrow \; x = \pm y \Rightarrow$$

$$\stackrel{(I)}{\Longrightarrow} \; 1 \pm 2y^2 = 0 \to \left(\frac{1}{2} = y^2 \wedge x = -y\right) \Rightarrow \pm\left[\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right].$$

$f\left(\pm\left[\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right]\right) = \pm\sqrt{\frac{1}{2}}$. Funkce $f$ ma tedy maximum $\sqrt{\frac{1}{2}}$ a nabyva ho pouze v bode $\left[\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right]$.

**9. priklad**

Zadani: $f(x,y) := xye^{-xy}$, $M := \{[x,y] \in \mathbb{R}^2;\; x \geq 0,\; y \geq 0\}$.

Reseni: Oznacme $t := xy \geq 0$, $g(t) := te^{-t}$. Pak zjevne $f(x,y) = g(t)$ a staci tedy hledat pouze extremy funkce $g$ na mnozine $\langle 0, +\infty)$. Protoze $g'(t) = (1-t)e^{-t}$, tak $g'(t) < 0$ na $(1,+\infty)$ a $g'(t) > 0$ na $(0,1)$. Protoze $g \in \mathcal{C}^1(\langle 0,+\infty))$ a $\lim_{t\to+\infty} g(t) = 0 = g(0)$, tak $g$ ma minimum $0$ v bode $0$ a maximum $\frac{1}{e}$ v bode $1$. Mame tedy, ze $f$ ma vzhledem k mnozine $M$ minimum $0$ a nabyva ho pouze na mnozine $\{[x,y] \in \mathbb{R}^2;\; xy = 0\}$ a $f$ ma vzhledem k mnozine $M$ maximum $\frac{1}{e}$ a nabyva ho pouze na mnozine $\{[x,y] \in \mathbb{R}^2;\; xy = 1\}$

**10. priklad**

Zadani: $f(x,y) := \operatorname{arccotg}(x^2+y)$, $M := \{[x,y] \in \mathbb{R}^2;\; 4y^3-4y+x^2 = 0,\; y > 0\}$.

Reseni: Protoze arccotg je klesajici funkce, staci hledat extremy funkce $g(x,y) := x^2+y$ a nasledne je dosadit do funkce arccotg.

(A) Oznacime $\Phi(x,y) := 4y^3-4y+x^2$, $\Psi(x,y) = y$. $g, \Phi, \Psi$ jsou polynomy a tedy $g, \Phi, \Psi \in \mathcal{C}^1(\mathbb{R}^2)$.

$$\nabla g(x,y) = [2x, 1], \quad \nabla\Phi(x,y) = [2x, 12y^2-4].$$

(B) Oznacme
$$N := \{[x,y] \in \mathbb{R}^2;\ 4y^3 - 4y + x^2 = 0,\ y \geq 0\}.$$
Ukazeme, ze $\overline{M} = N = M \cup \{\boldsymbol{o}\}$. $\Phi(x,0) = x^2 = 0$ prave tehdy kdyz $x = 0$. Mame tedy $N = M \cup \{\boldsymbol{o}\}$. $N = \Phi^{-1}(\{0\}) \cap \Psi^{-1}(\langle 0, +\infty)),\ \Phi, \Psi \in \mathcal{C} \overset{(V11)}{\Longrightarrow} N$ je uzavrena a tedy $N \supset \overline{M}$. Zbyva tedy ukazat, ze $\{\boldsymbol{o}\} \in \overline{M}$. Pro $y \in (0,1)$ mame $\Phi(0,y) < 0$. Pro $x \in \mathbb{R}$ mame $\Phi(x,0) > 0$. Necht $1 > \varepsilon > 0$ je libovolne, pak zvolim $0 < x, y < \varepsilon$ a mam tedy $[x,0], [0,y] \in B(\boldsymbol{o}, \varepsilon)$. Zvolime $h(t) := \Phi((1-t)x, ty)$. Pak $h \in \mathcal{C}(\mathbb{R})$, $h(0) = \Phi(x,0) > 0$ a $h(1) = \Phi(0,y) < 0$. Z Bolzanovy vety o nabyvani mezihodnot plyne, ze existuje $t_0 \in (0,1)$, ze $h(t_0) = \Phi((1-t_0)x, t_0 y) = 0$. Vzhledem k tomu, ze bod $[(1-t_0)x, t_0 y]$ lezi na usecce spojujici dva body lezici uvnitr koule (konvexni mnoziny), tak i tento bod lezi uvnitr teto koule. Mame tedy $B(\boldsymbol{o}, \varepsilon) \cap M \neq \emptyset$ pro kazde $\varepsilon > 0$. Pak ovsem $\boldsymbol{o} \in \overline{M}$.

(C) Z $\Phi(x,y) = 0$ plyne, ze $y^3 \leq y$ a tedy $y \leq 1$. Protoze $y \in \langle 0, 1\rangle$, tak $x^2 = 4y - 4y^3 \leq 4$. Pak $\overline{M} \subset \langle -2, 2\rangle \times \langle 0, 1\rangle$ a je tedy omezena.

(D) OK

(E) Mnozina $\overline{M} = M \cup \{\boldsymbol{o}\}$, kde $M = M_1$ je typu $\beta$.

(F$_1$) Lag. mult. (V24) pro vazbu $\Phi$.

(a) $\nabla \Phi(x,y) = \boldsymbol{o} \Leftrightarrow [x,y] = \left[0, \pm\sqrt{\frac{1}{3}}\right] \notin M$.

(b)

$$\begin{aligned}
(I): \quad 2x + 2\lambda x &= 0; \quad \text{pokud } x = 0 \overset{(III)}{\Longrightarrow} y \in \{0, \pm 1\} \Rightarrow \\
(II): \quad 1 + 4\lambda(3y^2 - 1) &= 0; \quad \Rightarrow [x,y] = [0,1], \\
(III): \quad 4y^3 - 4y + x^2 &= 0; \quad \text{pokud } x \neq 0 \overset{(I)}{\Longrightarrow} \lambda = -1 \Rightarrow \\
\overset{(II)}{\Longrightarrow} 12y^2 &= 5 \overset{y>0}{\Longrightarrow} y = \sqrt{\frac{5}{12}} \overset{(III)}{\Longrightarrow} [x,y] = \boxed{\left[\pm\sqrt{\frac{7}{3}\sqrt{\frac{5}{12}}}, \sqrt{\frac{5}{12}}\right]}.
\end{aligned}$$

(G) $g(0,0) = 0,\ g\left(\left[\pm\sqrt{\frac{7}{3}\sqrt{\frac{5}{12}}}, \sqrt{\frac{5}{12}}\right]\right) = \frac{10}{3}\sqrt{\frac{5}{12}},\ g(0,1) = 1.$

(H) Funkce $f$ nabyva minimum $\operatorname{arccotg}\left(\frac{10}{3}\sqrt{\frac{5}{12}}\right)$ na mnozine $M$ pouze v bodech $\left[\pm\sqrt{\frac{7}{3}\sqrt{\frac{5}{12}}}, \sqrt{\frac{5}{12}}\right]$. Protoze bod $\boldsymbol{o} \in \overline{M} \setminus M$, tak funkce $f$ nenabyva maximum na $M$ a $\sup_M(f) = \frac{\pi}{2}$.

**10. priklad - alternativni reseni**

Zadani: $f(x,y) := \operatorname{arccotg}(x^2 + y)$, $M := \{[x,y] \in \mathbb{R}^2;\ 4y^3 - 4y + x^2 = 0,\ y > 0\}$.

Reseni: Protoze arccotg je klesajici funkce, staci hledat extremy funkce $g(x,y) := x^2 + y$ a nasledne je dosadit do funkce arccotg. Podobne jako v (B) ukazeme, ze $\overline{M} = M \cup \{\boldsymbol{o}\} = N$ a podobne jako v (C) ukazeme, ze $y \in \langle 0, 1\rangle$. Dosadime vztah $\Phi(x,y) = 0 \Leftrightarrow x^2 = 4y - 4y^3$ do $g$ a dostaneme funkci $h(y) = 5y - 4y^3$, jejiz extremy budeme zkoumat na mnozine $\langle 0, 1\rangle$. Protoze $h'(y) = 5 - 12y^2$, tak $h'(y) > 0$ pro $y \in \left(0, \sqrt{\frac{5}{12}}\right)$, $h'(y) < 0$ pro $y \in \left(\sqrt{\frac{5}{12}}, 1\right)$ a $h \in \mathcal{C}(\mathbb{R})$, tak $h$ muze nabyvat extremu pouze v bodech $0, 1, \sqrt{\frac{5}{12}}$. Pak jiz staci dosadit a jsme hotovi.

**11. priklad**

Zadani: $f(x,y) := x - y$, $M := \{[x,y] \in \mathbb{R}^2;\ x^3 - y^3 + 3xy = 0,\ x > 0,\ y < 0\}$.

Reseni:

(A) Oznacime $\Phi(x,y) := x^3 - y^3 + 3xy$. $f, \Phi$ jsou polynomy a tedy $f, \Phi \in \mathcal{C}^1(\mathbb{R}^2)$.

$$\nabla f(x,y) = [1,-1], \quad \nabla\Phi(x,y) = [3x^2 + 3y, -3y^2 + 3x].$$

(B) Oznacme

$$N := \{[x,y] \in \mathbb{R}^2;\ \Phi(x,y) = 0,\ y \le 0,\ x \ge 0\}.$$

Ukazeme, ze $\overline{M} = N = M \cup \{\boldsymbol{o}\}$. $\Phi(x,0) = x^3 = 0$ prave tehdy kdyz $x = 0$. Podobne $\Phi(0,y) = -y^3 = 0$ prave tehdy kdyz $y = 0$. Mame tedy $N = M \cup \{\boldsymbol{o}\}$. $N = \Phi^{-1}(\{0\}) \cap \{[x,y] \in \mathbb{R}^2;\ x \ge 0,\ y \le 0\}$, $\Phi, x, y \in \mathcal{C}$ $\overset{(V11)}{\Longrightarrow}$ $N$ je uzavrena a tedy $N \supset \overline{M}$. Zbyva tedy ukazat, ze $\{\boldsymbol{o}\} \in \overline{M}$. Pro $x \in (0,1)$ mame $\Phi(x,-x) = 2x^3 - 3x^2 < 0$ a $\Phi(x,0) > 0$. Necht $1 > \varepsilon > 0$ je libovolne, pak zvolim $0 < x < \frac{\varepsilon}{\sqrt{2}}$ a mam tedy $[x,-x], [x,0] \in B(\boldsymbol{o},\varepsilon)$. Zvolime $h(t) := \Phi(x,-tx)$. Pak $h \in \mathcal{C}(\mathbb{R})$, $h(0) = \Phi(x,0) > 0$ a $h(1) = \Phi(x,-x) < 0$. Z Bolzanovy vety o nabyvani mezihodnot plyne, ze existuje $t_0 \in (0,1)$, ze $h(t_0) = \Phi(x,-t_0x) = 0$. Vzhledem k tomu, ze bod $[x,-t_0x]$ lezi na usecce spojujici dva body lezici uvnitr koule (konvexni mnoziny), tak i tento bod lezi uvnitr teto koule. Mame tedy $B(\boldsymbol{o},\varepsilon) \cap M \neq \emptyset$ pro kazde $\varepsilon > 0$. Pak ovsem $\boldsymbol{o} \in \overline{M}$.

(C) Dokazeme sporem, ze $x \le 3$ a $y \ge -3$. Nejprve predpokladejme, ze $x \ge -y$ a $x > 3$, pak

$$0 = \Phi(x,y) = x^3 - y^3 + 3xy \overset{(y\le 0)}{\ge} x^3 + 3xy \overset{(x \ge -y, x \ge 0)}{\ge} x^3 - 3x^2 \overset{(x>3)}{>} 0,$$

coz je spor. Obdobne se ukaze, ze $-y \ge x$ a $-y > 3$ vede ke sporu. Tim jsme dokazali, ze mnozina $\overline{M} \subset \langle 0,3\rangle \times \langle -3,0\rangle$ a tedy je omezena.

(D) OK

(E) Mnozina $\overline{M} = M \cup \{\boldsymbol{o}\}$, kde $M = M_1$ je typu $\beta$.

(F$_1$) Lag. mult. (V24) pro vazbu $\Phi$.

(a) $\nabla\Phi(x,y) = \boldsymbol{o} \Leftrightarrow ([x,y] = \boldsymbol{o}) \vee ([x,y] = [1,-1])$, $[1,-1] \notin M$.

$$\begin{aligned}
\frac{\partial\Phi}{\partial x}(x,y) = &\quad 3x^2 + 3y = 0,\\
\frac{\partial\Phi}{\partial y}(x,y) = &\quad -3y^2 + 3x = 0,\\
x\frac{\partial\Phi}{\partial x}(x.y) - y\frac{\partial\Phi}{\partial y}(x,y) = &\quad 3(x^3 + y^3) = 0 \Rightarrow x = -y,\\
\frac{\partial\Phi}{\partial x}(x,-x) = &\quad 3(x^2 - x) = 0 \Leftrightarrow x \in \{0,1\}.
\end{aligned}$$

(b)

$$\begin{aligned}
(I): &\quad 1 + 3\lambda(x^2 + y) = 0;\ (I) \Rightarrow \lambda \neq 0,\\
(II): &\quad -1 + 3\lambda(x - y^2) = 0;\\
(III): &\quad x^3 - y^3 + 3xy = 0;\\
\frac{(I)+(II)}{3\lambda}: &\quad x^2 - y^2 + x + y = (x+y)(x-y+1) = 0 \Rightarrow\\
&\overset{(x-y+1>0)}{\Longrightarrow} y = -x \overset{(III), x>0}{\Longrightarrow} x = \frac{3}{2} \implies [x,y] = \left[\frac{3}{2}, -\frac{3}{2}\right].
\end{aligned}$$

(G) $g(0,0) = 0$, $g\left(\frac{3}{2}, -\frac{3}{2}\right) = 3$.

(H) Funkce $f$ nabyva maximum 3 na mnozine $M$ pouze v bode $\left[\frac{3}{2}, -\frac{3}{2}\right]$. Protoze bod $\boldsymbol{o} \in \overline{M} \setminus M$, tak funkce $f$ nenabyva minimum na $M$ a $\inf_M(f) = 0$.